MODULARITY AND QUANTUM TOPOLOGY WORKSHOP SUMMARY

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1. Objectives

The workshop Modularity and Quantum Topology was held Oct. 12-16, 2026 at the American Institute of Mathematics (AIM) in Pasadena, CA. The goals of this workshop were to study emerging interactions between modular forms and quantum topology along with their connections to related areas such as representation theory, combinatorics, and mathematical physics. In particular, the workshop sought to investigate the following topics.

- Volume and Modularity Conjectures
- Habiro ring and colored Jones polynomials, identities, resurgence, and (holomorphic) quantum modularity
- The interplay among q-hypergeometric series, modular forms, representation theory, and combinatorics

2. Schedule

Monday Oct. 12, 2026

9:00am – Introduction

9:25am – Talk 1: Christine Lee, *Introduction to quantum topology*

10:55am – Break

11:10am – Talk 2: Kazuhiro Hikami,
Quantum modularity and invariants including WRT and CJP

12:25pm – Lunch

2:25pm – Problem Session (Moderator: Eleanor McSpirit, Scribe: Matthias Storzer)

4:00pm – Break

5:00pm – Happy Hour

Tuesday Oct. 13, 2026

9:00am – Talk 1: Veronica Fantini, *Resurgence*

10:30am – Break

11:00am – Talk 2: Josef Svoboda, *Quantum invariants including F_K and $\hat{Z}$*

12:20pm – Lunch

2:20pm – Project Groups

5:00pm – Happy Hour

6:00pm – Reception

Wednesday Oct. 14, 2026

9:00am – Talk 1: Shashank Kanade, *Vertex operator algebras*
10:25am – Break
11:00am – Talk 2: Kyungbae Park, *Modular knots*
12:00pm – Group Photo
12:10pm – Lunch
2:00pm – Project Groups
5:00pm – Happy Hour

Thursday Oct. 15, 2026

9:00am – Talk 1: Louisa Liles, *WRT invariants*
10:25am – Break
10:45am – Talk 2: Lara San Martín Suárez,
F_K for positive knots and tail of colored Jones polynomial
12:20pm – Lunch
2:20pm – Project Groups
5:00pm – Happy Hour

Friday Oct. 16, 2026

9:00am – Talk 1: Dongmin Gang, *VOA characters from 3-manifolds*
10:25am – Break + Announcements
11:15am – Talk 2: Yuya Murakami, *Quantum modularity of $\hat{Z}$-invariants*
12:25pm – Lunch
2:25pm – Project Groups
5:00pm – Happy Hour

3. Projects

3.1. **GPPV Invariants.** This working group discussed several questions about the Gukov-Pei-Putrov-Vafa (GPPV) invariants $\widehat{Z}$ of 3-manifolds, with an emphasis on extending them to supergroups. One of the main questions was how to define these invariants more systematically in the supergroup setting and how to understand the transformation $q \mapsto q^{-1}$. Closely related to this is the problem of defining $\widehat{Z}$ directly on the positive side, rather than treating it as a formal q-inversion of the usual series. The group also discussed possible connections between GPPV invariants on the positive side and logarithmic vertex operator algebras. Another direction of particular interest was the quantum modularity of the $\mathfrak{sl}(2|1)$ $\widehat{Z}$ invariants studied by Ferrari and Putrov. In this setting, one would like to understand both the quantum modular behavior of the associated q-series and the topological meaning

of their radial limits at roots of unity. The group reports progress on both questions. Going forward, the working group hopes to develop the supergroup construction further, find a definition of $\widehat{Z}$ on the positive side, and investigate its relation to $q \mapsto q^{-1}$ in concrete examples. The group also hopes to clarify how quantum modularity and possible connections with logarithmic vertex operator algebras fit into this picture.

This group includes Shashank Kanade, Jeremy Lovejoy, Eleanor McSpirit, Antun Milas, Yuya Murakami, and Oğuz Öner.

3.2. **Manx Knots.** It is known due to the work of Armond that "heads" and "tails" of colored Jones polynomials exist for alternating links and for adequate links. Garoufalidis–Le independently established this and also gave formulas for tails in terms of Nahm sums. A class of knots called "manx knots" was later investigated by Lee–van der Veen, and it was shown that the coefficient of the lowest term in q grows roughly as $n/2$ where n is the color. Thus, tails for this class of knots do not exist. However, it appears to the working group that if one renormalizes the nth colored Jones polynomial by the entire trailing monomial, then the large color limits may exist again. This phenomenon is related to, but different than the notion of "c-stability" introduced by Garoufalidis–Vuong.

During the workshop, the working group investigated this phenomenon for some examples of links. They found that this full renormalization procedure indeed helps to restore the large color limits in several examples, and that the limits are again essentially modular invariant (for the class of links the group explored). The group is currently working to prove their findings rigorously. The group will also explore related problems.

This group includes Shashank Kanade, Christine Ruey Shan Lee, Matthias Storzer, Leah Sturman, Lara San Martín Suárez, and Yi Wang.

3.3. **Explicit Methods and Exact Formulas for Quantum Invariants.** The working group on explicit methods and formulas discussed how quantum invariants for certain families of knots and 3-manifolds can be expressed as limiting values of Eichler integrals of half-integral weight theta functions. This has been worked out explicitly for the WRT invariant and the Kashaev invariant in a series of papers by Hikami. These Eichler integral expressions also arise naturally in the theory of q-hypergeometric series via the theory of Bailey pairs. The group discussed a recent paper of Lovejoy, Osburn, and Storzer, where this theory was applied to prove some conjectures of Hikami and to recover formulas for the unified WRT invariant for the Brieskorn manifolds $\sum(2,3,6p-1)$ and $\sum(2,3,6p+1)$. These results rely on Bailey pairs in the A-group of Slater's list along with some similar pairs from elsewhere in the literature.

For future work the group discussed the possibility of widening the net and looking at other Bailey pairs from Slater's list in the framework developed by Lovejoy-Osburn-Storzer. Presumably many of these would give rise to Eichler integrals, which might in turn match up with known expressions for quantum invariants. The working group considered specifically targeting spherical Seifert manifolds such as $M(2,2,K)$, where Eichler integral expressions for the WRT invariants remain relatively simple. The group also discussed the fact that if they end up finding q-hypergeometric formulas for the unified WRT invariants of some manifolds then it might be possible to "reverse engineer" and read off the cyclotomic coefficients of the colored Jones polynomials of the knots from which the manifolds are constructed via surgery.

This group includes Jeremy Lovejoy, Robert Osburn, and Leah Sturman.

3.4. **Modular Knots.** The modular knots working group brought together topological, arithmetic, and combinatorial perspectives on knots associated with primitive hyperbolic conjugacy classes in $\mathrm{PSL}_2(\mathbb{Z})$. During the workshop, the group discussed the Rademacher symbol's interpretation as a linking number with the missing trefoil, its relation to Dedekind sums, and its analogues for triangle groups. The group also examined connections with partition rank and compared descriptions of modular knots using hyperbolic matrices, Lorenz knots, and integer partitions, identifying potential applications to the study of Alexander polynomials.

These discussions motivated three directions for further collaboration. First, the group aims to investigate forms of imperfect quantum modularity associated with generalized Rademacher symbols for triangle groups, building on their topological interpretations and connections with discriminant-type automorphic forms. Possible higher analogues are also of interest. Second, the working group plans to use the relationships between partition rank, modular knots, and Dedekind sums to obtain explicit formulas and asymptotic results for Dedekind sums associated with families of partitions. The group will also explore whether the knot-theoretic interpretation of rank leads to a suitable notion of quantum modularity for the statistic itself. Finally, the group is interested in studying Alexander polynomials of modular knots using partitions and related combinatorial methods, seeking improved bounds on the number of distinct polynomials in suitably bounded families.

This group includes Amanda Folsom, Soon-Yi Kang, Louisa Liles, Jeremy Lovejoy, Robert Osburn, Kyungbae Park, Leah Sturman, Holly Swisher, and Yi Wang.

3.5. **Non-modularity of Tails.** It is known that the tail Φ_K of the coloured Jones polynomial for some alternating knots K can be written as a product of (partial) theta functions and thus has modular properties. However, for other knots like 8_5, asymptotics suggest that Φ_K is not modular at all. The objective of this project was to understand the topological meaning of the non-modularity of Φ_K for alternating knots. First experiments suggest that the (non)modularity of Φ_K for a given knot K is closely related to the hyperbolic structure of the manifold (or, more generally, orbifold) obtained by (generalised) Dehn surgery on K. This working group aims to make this connection more precise by considering more examples and classes of knots, such as double-twist knots.

This group includes Lara San Martín Suárez, Matthias Storzer, Josef Svoboda, and Yi Wang.

3.6. **Resurgence.** The resurgence group spent the week investigating the relationship between resurgence and quantum modularity. Starting with the example of the figure-eight knot 4_1, the group asked whether quantum modularity was equivalent to the effectiveness of median resummation and, in particular, whether the median resummation of the associated asymptotic series recovers the quantum modular cocycle. The group also studied a q-series whose associated L-function is a product of shifted Riemann zeta functions and proved that its asymptotic expansion is modular resurgent. Several members of the group are continuing to study this example.

The group then considered the q-series obtained by taking the Dirichlet coefficients of a Dedekind zeta function as the coefficients of a generating function. For real quadratic fields, the group found that the resulting q-series is a holomorphic quantum modular form and that

its asymptotic expansion is modular resurgent. The group has begun to investigate whether analogous results hold for Hecke L-functions of real quadratic fields. It also considered totally real fields of higher degree, beginning with the cubic case. In these examples the asymptotic series are no longer Gevrey-1, and the group is investigating whether modified Borel and Laplace transforms or other normalizations of the q-series can be used to associate quantum modular forms to these Dedekind zeta functions.

This group includes Veronica Fantini, Rajat Gupta, Gene Kopp, Eleanor McSpirit, Yuya Murakami, Oğuz Öner, Olav Richter, and Josef Svoboda.

3.7. **VOAs and Knots.** This project explores a possible connection between three-manifolds with torus boundary and characters of rational vertex operator algebras (VOAs). Starting from an ideal triangulation, the project investigates how its Neumann–Zagier data can be used to construct candidate q-series of Nahm-sum type. A central question is when these series, perhaps after a suitable normalization, can be identified with characters of rational VOAs. The working group is particularly interested in the geometric or topological features of the three-manifold that may account for the appearance of rationality and modular behavior.

The group plans to investigate two families: complements of torus knots $T(P, Q)$ in S^3, with P and Q coprime, and mapping tori of the once-punctured torus, specified by monodromies in $SL(2, \mathbb{Z})$ up to conjugacy. For each family, the working group hopes to determine which rational VOA characters can arise and how they depend on the knot parameters or the monodromy. The working group also plans to explore whether the construction can be extended to link complements, with multiple torus boundary components, and to closed three-manifolds.

This group includes John Duncan, Dongmin Gang, Kazuhiro Hikami, and Shashank Kanade.